AP Calculus BC

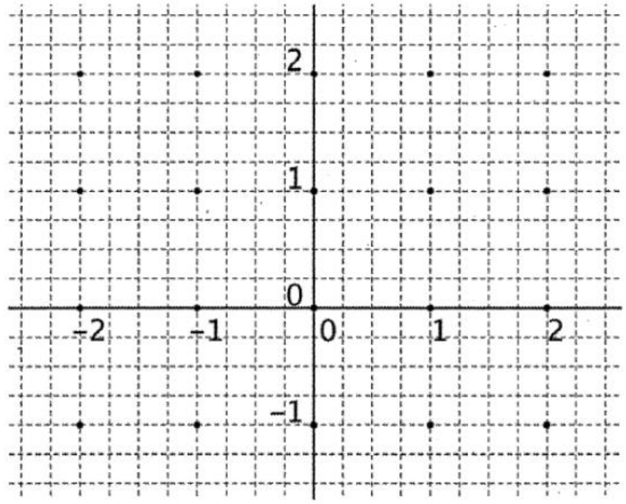
Unit 12: Differential Equations

AP Calculus BC – Worksheet 97 **Differential Equations and Slope Fields**

Sketch the slope field for the given differential equations at the indicated points.

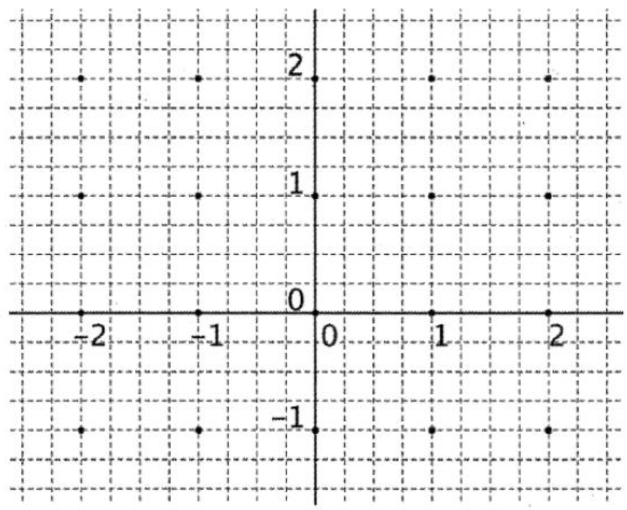
1. $\frac{dy}{dx} = x + y$

Point	Slope	Point	Slope
(-2, -1)		(0, 1)	
(-2, 0)		(0, 2)	
(-2, 1)		(1, -1)	
(-2, 2)		(1, 0)	
(-1, -1)		(1, 1)	
(-1, 0)		(1, 2)	
(-1, 1)		(2, -1)	
(-1, 2)		(2, 0)	
(0, -1)		(2, 1)	
(0, 0)		(2, 2)	



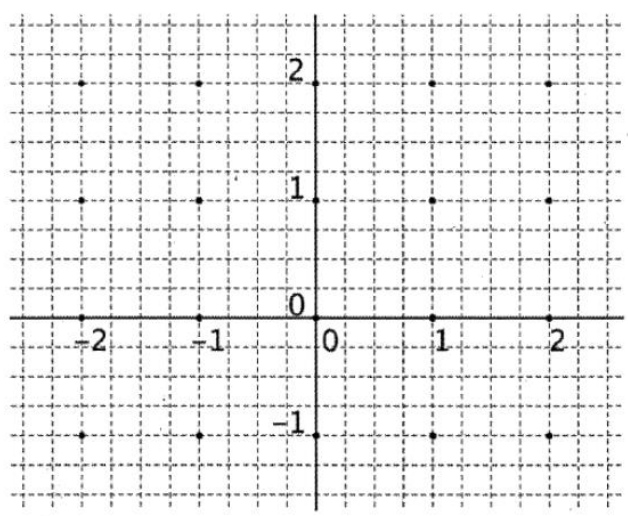
2. $\frac{dy}{dx} = xy$

Point	Slope	Point	Slope
(-2, -1)		(0, 1)	
(-2, 0)		(0, 2)	
(-2, 1)		(1, -1)	
(-2, 2)		(1, 0)	
(-1, -1)		(1, 1)	
(-1, 0)		(1, 2)	
(-1, 1)		(2, -1)	
(-1, 2)		(2, 0)	
(0, -1)		(2, 1)	
(0, 0)		(2, 2)	



3. $\frac{dy}{dx} = {}^{-y}\!/_{x}$

Point	Slope	Point	Slope
(-2, -1)		(0, 1)	
(-2, 0)		(0, 2)	
(-2, 1)		(1, -1)	
(-2, 2)		(1, 0)	
(-1, -1)		(1, 1)	
(-1, 0)		(1, 2)	
(-1, 1)		(2, -1)	
(-1, 2)		(2, 0)	
(0, -1)		(2, 1)	
(0, 0)		(2, 2)	



Match the differential equations with the appropriate slope field. For each graph sketch the particular solution that passes through the point $(0,1)$.

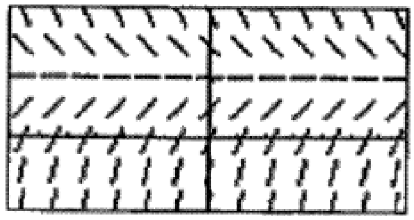
4. $\dfrac{dy}{dx} = \sin x$ _____

5. $\dfrac{dy}{dx} = x - y$ _____

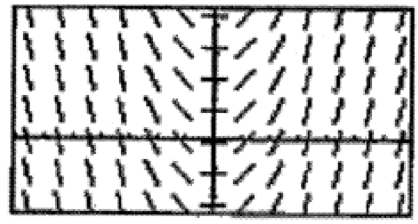
6. $\dfrac{dy}{dx} = 2 - y$ _____

7. $\dfrac{dy}{dx} = x$ _____

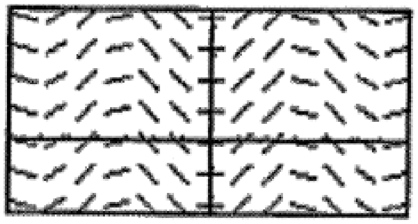
(A)



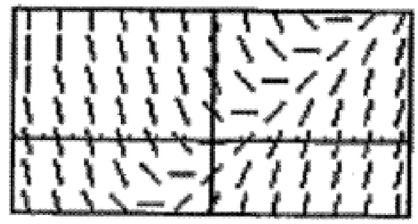
(B)



(C)



(D)



Match the differential equations with the appropriate slope field. For each graph sketch the particular solution that passes through the point $(-1,-1)$.

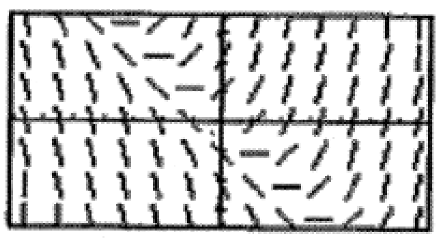
8. $\dfrac{dy}{dx} = 0.5x - 1$ _____

9. $\dfrac{dy}{dx} = 0.5y$ _____

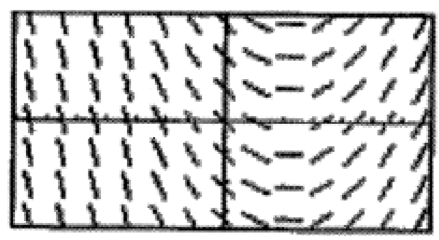
10. $\dfrac{dy}{dx} = -\dfrac{x}{y}$ _____

11. $\dfrac{dy}{dx} = x + y$ _____

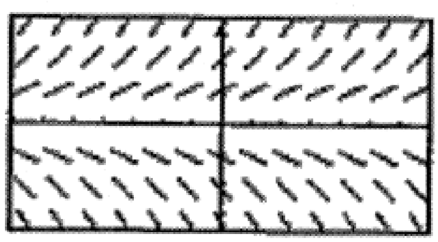
(A)



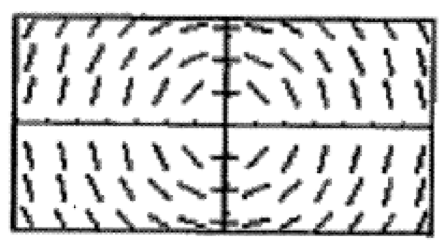
(B)



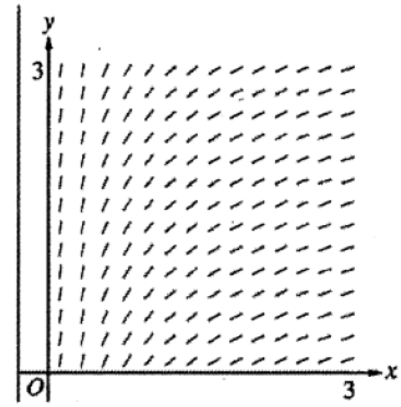
(C)



(D)



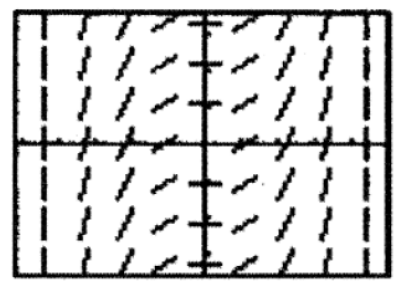
12. At the right is a slope field from a certain differential equation. Which of the following could be a specific solution to that differential equation?



(A) $y = x^2$ (B) $y = e^x$ (C) $y = e^{-x}$

(D) $y = \cos x$ (E) $y = \ln x$

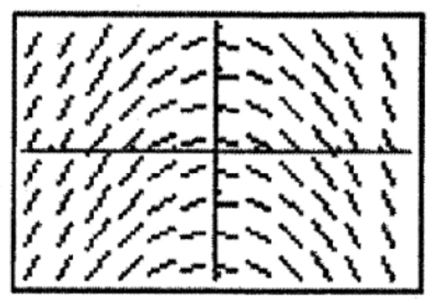
13. At the right is a slope field from a certain differential equation. Which of the following could be a specific solution to that differential equation?



(A) $y = \sin x$ (B) $y = \cos x$ (C) $y = x^2$

(D) $y = \frac{1}{6}x^3$ (E) $y = \ln x$

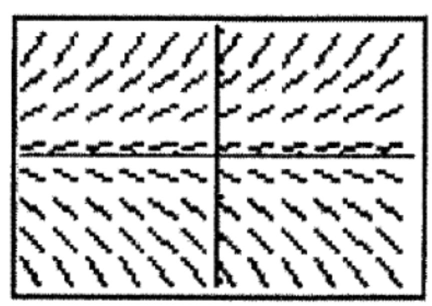
14. At the right is a slope field from a certain differential equation. Which of the following could be a specific solution to that differential equation?



(A) $y = \sin x$ (B) $y = \cos x$ (C) $y = -x^2$

(D) $y = \tan x$ (E) $y = e^{-x}$

15. At the right is a slope field from a certain differential equation. Which of the following could be a specific solution to that differential equation?



(A) $y = x^2$ (B) $y = e^x$ (C) $y = e^{-x}$

(D) $x = y^2$ (E) $y = \ln x$

Use Euler's Method with increments $\Delta x = 0.1$ to approximate the value of y when $x = 1.3$

1. $\frac{dy}{dx} = x - 1$ and $y = 2$ when $x = 1$	2. $\frac{dy}{dx} = 2x - y$ and $y = 0$ when $x = 1$

Use Euler's Method with increments $\Delta x = -0.1$ to approximate the value of y when $x = 1.7$

3. $\frac{dy}{dx} = 2 - x$ and $y = 1$ when $x = 2$	4. $\frac{dy}{dx} = x - y$ and $y = 2$ when $x = 2$

5. Given the differential equation $\frac{dy}{dx} = x - 2$ and $y(0) = 5$.

(a) Find an approximation for $y(0.8)$ by using Euler's method with two equal steps. Sketch your solution.

6. If $\frac{dy}{dx} = 2x - y$ and if $y = 3$ when $x = 2$, use Euler's method with five equal steps to approximate y when $x = 1.5$.

7. Assume that f and f' have the values given in the table. Use Euler's method with two equal steps to approximate the value of $f(2.6)$.

x	3	2.8	2.6
$f'(x)$	0.4	0.7	0.9
$f(x)$	2		

AP Multiple Choice

x	$f'(x)$
1	0.2
1.5	0.5
2	0.9

The table above gives values of f', the derivative of a function f. If $f(1) = 4$, what is the approximation to $f(2)$ obtained by using Euler's method with a step size of 0.5 ?

(A) 2.35 (B) 3.65 (C) 4.35 (D) 4.70 (E) 4.80

Let $y = f(x)$ be the solution to the differential equation $\frac{dy}{dx} = 2y - x$ with initial condition $f(1) = 2$. What is the approximation for $f(0)$ obtained by using Euler's method with two steps of equal length starting at $x = 1$?

(A) $-\frac{5}{4}$ (B) -1 (C) $\frac{1}{4}$ (D) $\frac{1}{2}$ (E) $\frac{27}{4}$

1

Given the differential equation $\frac{dy}{dx} = x + 2$ with initial condition $y(0) = 3$.

(a) Find the approximation for $y(1)$ using Euler's method with two equal steps.

(b) Solve the differential equation $\frac{dy}{dx} = x + 2$ with initial condition $y(0) = 3$. Use your solution to find $y(1)$.

(c) The error in using Euler's Method is the difference between the approximate value and the exact value. What is the error in the approximation found in (a). How could you produce a smaller error using Euler's Method?

2

Suppose a continuous function *f* and its derivative f' have values that are given in the table below.

x	2.0	2.5	3.0
$f'(x)$	0.4	0.6	0.8
$f(x)$	5		

Use Euler's Method with two steps of equal size to approximate the value of $f(3)$.

3

Given the differential equation $\frac{dy}{dx} = \frac{1}{x+2}$ and $y(0) = 1$, find an approximation of $y(1)$ using Euler's Method with two steps of size $\Delta x = 0.5$.

4

Given the differential equation $\frac{dy}{dx} = x + y$ and $y(1) = 3$, find an approximation of $y(2)$ using Euler's Method with two equal steps.

5

The curve passing through $(2,0)$ satisfies the differential equation $\frac{dy}{dx} = 4x + y$. Find an approximation to $y(3)$ using Euler's Method with two equal steps.

6

Suppose a continuous function *f* and its derivative f' have values that are given in the table below.

x	4	4.2	4.4
$f'(x)$	-0.5	-0.3	-0.1
$f(x)$	2		

Use Euler's Method with two steps of equal size to approximate the value of $f(4.4)$.

7

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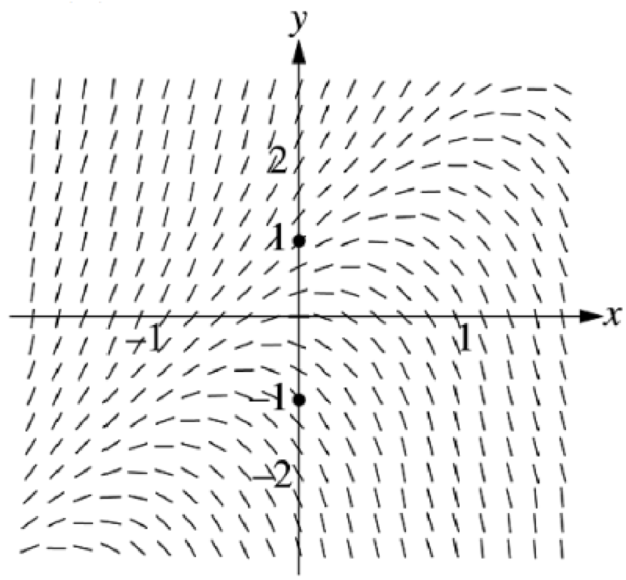
Consider the differential equation: $\frac{dy}{dx} = 2y - 4x$

(a) The slope field for the given differential equation is provided. Sketch the solution curve that passes through the point $(0,1)$ and sketch the solution curve that passes through the point $(0,-1)$.\



(b) Let f be the function that satisfies the given differential equation the initial condition $f(0) = 1$. Use Euler's Method, starting at $x = 0$ with a step size of 0.1, to approximate $f(0.2)$. Show that work that leads to your answer.

(c) Find the value of b for which $y = 2x + b$ is a solution to the given differential equation. Justify your answer.

(d) Let g be the function that satisfies the given differential equation with initial condition $g(0) = 0$. Does the graph of g have a local extremum at the point $(0,0)$. If so, is the point a local maximum or a local minimum? Justify your answer.

AP Calculus BC – Worksheet 100 **Separable Differential Equations and Exponential Growth**

For problems 1-10, find the particular solution to the differential equation with the given initial condition.
*- optional problems

1) $\frac{dy}{dx}=\frac{x}{y},\ \ y(1)=-2$	2) $\frac{dy}{dx}=-\frac{x}{y},\ \ y(4)=3$
3) $\frac{dy}{dx}=\frac{y}{x},\ \ y(2)=2$	4) $\frac{dy}{dx}=2xy,\ \ y(0)=-3$
5) $\frac{dy}{dx}=xy+5x+2y+10,\ \ y(0)=-1$	6) $\frac{dy}{dx}=\cos^2 y,\ \ y(0)=0$
7) $\frac{dy}{dx}=-2xy^2,\ \ y(1)=0.25$	8) $\frac{dy}{dx}=\frac{4\sqrt{y}\ln x}{x},\ \ y(e)=1$
*9) $\frac{dy}{dx}=e^{x-y},\ \ y(0)=2$	*10) $(\sec x)\frac{dy}{dx}=e^{y+\sin x},\ \ y(0)=0$

11	A sampling of a certain radioactive isotope loses 99% of its radioactive matter in 199 hours. What is the half-life of the isotope?
12	Population y grows according to the equation $\frac{dy}{dt}=ky$, where k is a constant and t is measured in years. If the population doubles every 10 years, what is the value of k?
13	Bacteria in a certain culture increase at a rate proportional to the number present. If the number of bacteria doubles in three hours, in how many hours will the number of bacteria triple?
14	A puppy weighs 2.0 pounds at birth and 3.5 pounds two months later. If the weight of the puppy during the first 6 months is increasing at a rate proportional to its weight, how much will the puppy weigh at 3 months old?
15	During a certain epidemic, the number of people that are infected at any time increases at a rate proportional to the number of people that are infected at that time. If 1,000 people are infected when the epidemic is first discovered, and 1,200 are infected 7 days later, how many people are infected 12 days after the epidemic is first discovered?
16	If $\frac{dy}{dt}=-2y$ and $y=1$ when $t=0$, what is the value of t for which $y=\frac{1}{2}$?

AP Calculus BC – Worksheet 101 **Logistic Growth**

Show all work. No calculator unless stated

Multiple Choice

1	The spread of a disease through a community can be modeled by the logistic equation $y = \dfrac{600}{1+59e^{-0.1t}}$, where *y* is the number of people infected after *t* days. How many people are infected when the disease is spreadin the fastest?
2	The spread of a disease through a community can be modeled by the logistic equation $y = \dfrac{0.9}{1+45e^{-0.15t}}$, where *y* is the **proportion** of people infected after *t* days. According to the model, what percentage of people in the community will not become infected?
3	The population $P(t)$ of a species satisfies the logistic differential equation $\dfrac{dP}{dt} = P\left(2 - \dfrac{P}{5000}\right)$, where the initial population is $P(0) = 3000$ and time *t* is measured in years. What is $\lim\limits_{t\to\infty} P(t)$?
4	Suppose a population of wolves grows according to the logistic differential equation $\dfrac{dP}{dt} = 3P - 0.01P^2$, where *P* is number of wolves at time *t*, in years. Which of the following statements is true? I. $\lim\limits_{t\to\infty} P(t) = 300$ II. The growth rate of the wolf population is greatest when $P = 150$. III. If $P > 300$, the population of wolves is increasing.

5	Suppose the population of bears in a national park grows according to the differential equation $\dfrac{dP}{dt} = 5P - 0.002P^2$, where *P* is the number of bears at time *t*, in years. (a) If $P(0) = 100$, find $\lim\limits_{t\to\infty} P(t)$. Sketch a possible graph for $P(t)$. For what values of *P* is the graph of *P* increasing? Decreasing? Justify your answer. (b) If $P(0) = 1500$, find $\lim\limits_{t\to\infty} P(t)$. Sketch a possible graph for $P(t)$. For what values of *P* is the graph of *P* increasing? Decreasing? Justify your answer. (c) If $P(0) = 3000$, find $\lim\limits_{t\to\infty} P(t)$. Sketch a possible graph for $P(t)$. For what values of *P* is the graph of *P* increasing? Decreasing? Justify your answer. (d) How many bears are in the park when the population of bears is growing the fastest? Justify your answer.

6

A population of animals is modeled by a function P that satisfies the logistic differential equation $\frac{dP}{dt} = 0.01P(100 - P)$, where t is measured in years.

(a) If $P(0) = 20$, solve for P as a function of t.

(b) Use your answer in (a) to find P when $t = 3$ years.

(c) Use your answer in (a) to find t when $P = 80$ animals.

7

The rate at which a rumor spreads through a high school of 2000 students can be modeled by the differential equation $\frac{dP}{dt} = 0.003P(2000 - P)$, where P is the number of students who have heard the rumor t hour after 9:00AM.

(a) How many students have heard the rumor when it is spreading the fastest?

(b) If $P(0) = 5$, solve for P as a function of t.

(c) Use your answer in (b) to determine how many hours have passed when the rumor is spreading the fastest.

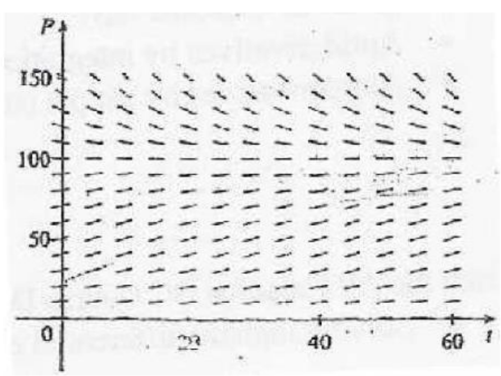
(d) Use your answer in (b) to determine the number of people who have heard the rumor after two hours.

8

Suppose that a population develops according to the logistic equation $\frac{dP}{dt} = 0.05P - 0.0005P^2$ where t is measured in weeks.

(a) What is the carrying capacity limit to growth?

(b) A slope field for this equation is shown below:



i) For what values of P is $\frac{dP}{dt}$ close to zero?

ii) For what value of P is $\frac{dP}{dt}$ greatest?

iii) For what values of P is $\frac{dP}{dt}$ negative?

iv) For what values of P is $\frac{dP}{dt}$ positive?

(c) Use the slope field to sketch solutions for initial populations of 20, 60, 120.

i) What do these solutions have in common? How do the solutions differ?

ii) Which solutions have inflection points?

iii) At what population level do these inflection points occur?

9

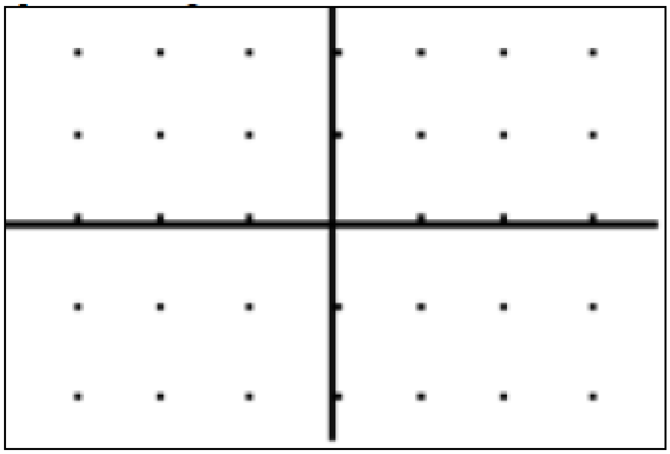
Newton's Law of Cooling states that the rate of change in the temperature of an object is proportional to the difference between the object's temperature and the temperature of the surrounding environment. A detective finds a murder victim at 9AM. The temperature of the body is measured at $90.3°F$. One hour later, the temperature of the body is $89°F$. The temperature of the room has been maintained at a constant of $68°F$.

(a) Assuming the temperature, T, of the body obeys Newton's Law of Cooling, write a differential equation for T, in degrees Fahrenheit, as a function of t hours.

(b) Solve the differential equation and use the solution to estimate when the murder occurred.

AP Calculus BC – Worksheet 102 **Differential Equations Review**

1	At any time $t \geq 0$, in days, the rate of growth of a bacteria population is given by $\frac{dy}{dt} = ky$, where *k* is a constant and *y* is the number of bacteria present. The initial population is 1,000 and the population triples during the first 5 days. (a) What an expression for *y* at any time $t \geq 0$. (b) By what factor will the population have increased in the first 10 days? (c) At what time *t*, in days, will the population have increased by a factor of 6?
2	In a community of 45,000, the rate of growth of a flu epidemic satisfies the logistic differential equation $\frac{dP}{dt} = kP\left(45{,}000 - P\right)$, where *P* is the number of people who have the disease at time *t*, in weeks. Two hundred people have the flu at the outset. Twenty eight hundred people have it after 3 weeks. (a) How many people have the disease after 5 weeks? (b) When is the disease growing the fastest?
3	Suppose that the grown of a population, *y*, is given by the logistic equation $y = \frac{1000}{1+999e^{-0.9t}}$. (a) What is the population at time $t = 0$? (b) What is the carrying capacity? (c) What is the constant *k*? (d) When does the population reach 75% of the carrying capacity?
4	The population of a country increases at a rate proportional to the existing population. If the population doubles in 20 years, determine the constant of proportionality.
5	If a substance decomposes at a rate proportional to the amount of the substance present, and if the amount decreases from 40 g to 10 g in 2 hours, determine the constant of proportionality.
6	Solve the separable differential equation $\frac{dy}{dx} = \frac{\sin x}{3y^2}$ where $y(\pi) = 5$.
7	A population of moose in a wildlife preserve is modeled by a function *P* that satisfies the logistic differential equation $\frac{dP}{dt} = 0.0425P(40 - P)$. a) If $P(0) = 4$, what is $\lim_{t\to\infty} P(t)$? b) If $P(0) = 60$, what is $\lim_{t\to\infty} P(t)$? c) If $P(0) = 4$, for what value of *P* is the population growing at the fastest rate? d) Given that $P(0) = 4$, write an equation *P(t)* that gives the population for any time *t*.

9	Consider the differential equation $\frac{dy}{dx} = 2x - y$. a) On the axes below, sketch a slope field for the given differential equation at the indicated points and sketch the solution curve that passes through the (0, 1).  [-3, 3] by [-2, 2] b) The solution curve that passes through (0, 1) has a local minimum at $x = \ln\left(\frac{3}{2}\right)$. What is the *y*-coordinate of this local minimum?
10	Use Euler's Method with increments $\Delta x = 0.1$ to approximate the value of y when $x = 1.3$ for $\frac{dy}{dx} = x - 1$ and $y = 2$ when $x = 1$

NO CALCULATOR IS ALLOWED FOR THIS QUESTION.

1. The population of a virus in a host can be modeled by the function A that satisfies the differential equation $\frac{dA}{dt} = t - \frac{1}{5}A$, where A is measured in millions of virus cells and t is measured in days for $5 \le t \le 10$. At time $t = 5$ days, there are 10 million cells of the virus in the host.

(a) Write an equation for the line tangent to the graph of A at $t = 5$. Use the tangent line to approximate the number of virus cells in the host, in millions, at time $t = 7$ days.

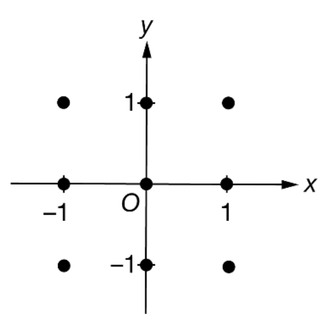
(b) Show that $A(t) = -25 + 5t + 10e^{-\frac{t}{5}+1}$ satisfies the differential equation $\frac{dA}{dt} = t - \frac{1}{5}A$ with initial condition $A(5) = 10$.

(c) The host receives an antiviral medication. The amount of medication in the host is modeled by the function M that satisfies the differential equation $\frac{dM}{dt} = -\frac{1}{2}\left(\frac{M}{t+k}\right)$, where M is measured in milligrams, t is measured in days since the host received the medication, and k is a positive constant. If the amount of medication in the host is 30 milligrams at time $t = 0$ days and 15 milligrams at time $t = 3$ days, what is $M(t)$ in terms of t?

NO CALCULATOR IS ALLOWED FOR THIS QUESTION.

2. Consider the differential equation $\frac{dy}{dx} = xy^4$.

(a) One the axes provided, sketch a slope field for the given differential equation at the nine points indicated.



(b) Find $\frac{d^2y}{dx^2}$ in terms of x and y. Determine the concavity of all solution curves for the given differential equation in Quadrant II. Give a reason for your answer.

(c) Find the particular solution $y = f(x)$ to the given differential equation with initial condition $f(4) = -1$.

NO CALCULATOR IS ALLOWED FOR THIS QUESTION.

3. Liquid fertilizer is injected into a hydroponics growing system via a pumping system. The total amount of liquid fertilizer injected into the growing system by time t is modeled by the function F that satisfies the logistic differential equation $\frac{dF}{dt} = \frac{1}{3}F(6-F)$, where t is measured in months and F is measured in liters. At time $t=0$, 3 liters of liquid fertilizer are injected into the growing system. (Note: Hydroponics is the process of growing plants in sand, gravel, or liquid, with added nutrients but without soil.)

(a) Find $\lim_{t\to\infty} F(t)$ and $\lim_{t\to\infty} \frac{dF}{dt}$.

(b) Find the value of $\frac{dF}{dt}$ at the time when F is increasing most rapidly. Give a reason for your answer and indicate units of measure.

(c) Find $\frac{d^2F}{dt^2}$ in terms of F.

(d) Use Euler's method, starting at $t=0$ with two steps of equal size, to approximate the total amount of liquid fertilizer injected into the growing system by time $t=1$ month. Show the computations that lead to your answer. Is the approximation an overestimate or an underestimate for the total amount of liquid fertilizer injected into the growing system by time $t=1$ month? Give a reason for your answer.